. MEAN VALUE THEOREMS

In this section we shall study Rolle's theorem, Lagrange's theorem and Cauchy's mean eorems and their geometrical interpretations. Also we shall prove Taylor's theorem which is neral form of Lagrange's mean value theorem when the function is differentiable n times.

1. ROLLE'S THEOREM

Let f be a function defined on $[a, b]$ such that

(i) f is continuous on $[a, b]$

(ii) f is derivable on (a, b)

(iii) $f(a) = f(b)$,

hen there exists atleast one real number c lying between a and b such that $f'(c) = 0$.

[K.U. 2016, 15, 13; M.D.U. 2015, 14, 06, 05]

Proof. Since the function f is continuous on $[a, b]$, therefore f is bounded on $[a, b]$ and ttain its bounds *i.e.*, there exists $c, d \in [a, b]$ such that $f(c) = M$ and $f(d) = m$, where M and m re Sup. f and Inf. f respectively.

Two different cases arise :

Case I. $M = m$ *i.e.*, *Sup.* f = *Inf.* f :

Here f is a constant function over $[a, b]$ and consequently $f'(x) = 0$ *for all* $x \in [a, b]$.

Hence $f'(c) = 0$ where $c \in (a, b)$.

Case II. $M \neq m$:

As $f(a) = f(b)$, therefore either M or m is different from $f(a) = f(b)$. Suppose that $M \neq f(a)$ and $M \neq f(b)$.

As $f(c) = M$, therefore c is different from both a and b and c lies in the open interval (a, b).

Since $f(c)$ is the supremum of f on $[a, b]$, therefore

$$f(x) \leq f(c) \text{ for all } x \in [a, b] \qquad ...(1)$$

From (1) $\quad f(c-h) \leq f(c)$

$\therefore \quad f(c-h) - f(c) \leq 0$

Dividing by $-h < 0$, we get

$$\frac{f(c-h) - f(c)}{-h} \geq 0$$

From (1), $\quad f(c+h) \leq f(c)$

$\therefore \quad f(c+h) - f(c) \leq 0$

Dividing by $h > 0$, we get

$$\frac{f(c+h) - f(c)}{h} \leq 0$$

Second Method :

(*i*) If $f(x)$ is a polynomial function of x, *i.e.*, of the type , $ax^3 + bx^2 + cx + d$, $3x^2 - 2x + 1$ etc., then it is continuous for all real x (*since polynomial functions of x are* **always** *continuous for real x*).

(*ii*) If $f(x)$ is not a polynomial function, but of the type $x + \frac{1}{x}$, $\sqrt{x^2 - 4}$, $\sin x$, $\log x$ etc. then we find $f'(x)$. If $f'(x)$ is finite, definite and real in $[a, b]$, then $f(x)$ is derivable in $[a, b]$ and hence continuous for all x in $[a, b]$ (*Since every derivable function is continuous also*).

2.8.2. How to check the derivability of $f(x)$ in (a, b)

(*i*) If $f(x)$ is a polynomial function of x, then $f(x)$ is derivable in (a, b) as a polynomial function is always derivable for all real values of x.

(*ii*) If $f(x)$ is not a polynomial function, we differentiate $f(x)$ with respect to x. If $f'(x)$ is finite, definite and real in (a, b), then $f(x)$ is derivable in (a, b).

Note.

1. If a function is continuous in a closed interval $[a, b]$, it is also continuous in open interval (a, b).
2. If a function is derivable in a closed interval $[a, b]$, it is also derivable in open interval (a, b).
3. Polynomial functions of the type $a_0 + a_1 x ....+ a_n x^n ....$ are always continuous and derivable.
4. Every derivable function is continuous also.
5. Sum, difference and product of two continuous functions is always continuous.

SOLVED EXAMPLES

Example 1. ***Prove that Rolle's theorem holds for the following functions:***

(*i*) $f(x) = x^2 - 5x + 6$ *in the interval* $[2, 3]$

(*ii*) $f(x) = x^3 - 7x^2 + 16x - 12$ *in the interval* $[2, 3]$

(*iii*) $f(x) = x(x-4)^2$ *in the interval* $[0, 4]$. [K.U. 2018]

Solution. (*i*) Here $f(x) = x^2 - 5x + 6$

$f(x)$ is a polynomial function of x and is therefore continuous and derivable for all real x.

Thus

(*a*) $f(x)$ is continuous in $[2, 3]$

(*b*) $f(x)$ is derivable in $(2, 3)$

(*c*) $f(2) = (2)^2 - 5(2) + 6 = 4 - 10 + 6 = 0$

and $f(3) = (3)^2 - 5(3) + 6 = 9 - 15 + 6 = 0$

$\therefore$ $f(2) = f(3)$

Thus all the conditions of Rolle's theorem are satisfied. Hence there must exist atleast one point $x = c$ in $(2, 3)$ such that $f'(c) = 0$.

Differentiating (1) w.r.t. x, we have

$$f'(x) = 2x - 5 \quad \Rightarrow \quad f'(c) = 2c - 5$$

Now, $f'(c) = 0$

$$\Rightarrow \quad 2c - 5 = 0 \quad \Rightarrow \quad c = \frac{5}{2} \in (2, 3)$$

Hence Rolle's theorem holds for the function $f(x) = x^2 - 5x + 6$ in $[2, 3]$.

(*ii*) Here $f(x) = x^3 - 7x^2 + 16x - 12$

(*a*) $f(x)$ being a polynomial function is continuous everywhere and hence is continuous in $[2, 3]$.

(*b*) $f'(x) = 3x^2 - 14x + 16$ which exists for all $x \in (2, 3)$

(*c*) $f(2) = (2)^3 - 7(2)^2 + 16(2) - 12 = 8 - 28 + 32 - 12 = 0$

$f(3) = (3)^3 - 7(3)^2 + 16(3) - 12 = 27 - 63 + 48 - 12 = 0$

$\therefore$ $f(2) = f(3)$

Thus all the three conditions of Rolle's theorem are satisfied. Hence there must exist atleast one value of $c \in (2, 3)$ such that $f'(c) = 0$

We have $f'(x) = 3x^2 - 14x + 16$

$\Rightarrow$ $f'(c) = 3c^2 - 14c + 16$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 14c + 16 = 0$

$$\Rightarrow \quad (c - 2)(3c - 8) = 0 \quad \Rightarrow \quad c = 2, \frac{8}{3}$$

$$c = \frac{8}{3} \in (2, 3)$$

Hence Rolle's theorem is verified.

(*iii*) Here $f(x) = x(x - 4)^2 = x^3 - 8x^2 + 16x$

(*a*) $f(x)$ being a polynomial is continuous in $[0, 4]$

(*b*) $f'(x) = 3x^2 - 16x + 16$ which exists for all $x \in (0, 4)$

(*c*) $f(0) = 0$ and $f(4) = 0 \Rightarrow f(0) = f(4)$

Thus all the three conditions of Rolle's theorem are satisfied. Hence there must exist atleast one value of $c \in (0, 4)$ such that $f'(c) = 0$

We have $f'(x) = 3x^2 - 16x + 16 \Rightarrow f'(c) = 3c^2 - 16c + 16$

Now, $f'(c) = 0 \Rightarrow 3c^2 - 16c + 16 = 0$

$$\Rightarrow \quad (c-4)(3c-4) = 0 \quad \Rightarrow \quad c = 4, \frac{4}{3}$$

$$c = \frac{4}{3} \in (0, 4).$$

Hence Rolle's theorem is verified.

Example 2. ***Verify Rolle's theorem for the function $f(x) = (x-a)^m (x-b)^n$, in the interval $[a, b]$, where m and n are positive integers.*** [C.D.L.U. 2016; K.U. 2013, 1

Solution. Here $f(x) = (x-a)^m (x-b)^n$

(*a*) Expanding $(x-a)^m$ and $(x-b)^n$ by binomial theorem, we observe that $f(x)$ is a polynomial of degree $(m+n)$.

Thus $f(x)$ being a polynomial is continuous everywhere and hence is continuous in $[a, b]$

(*b*) $f'(x) = (x-a)^m . n(x-b)^{n-1} + (x-b)^n . m(x-a)^{m-1}$

$$= (x-a)^{m-1}(x-b)^{n-1}[n(x-a) + m(x-b)]$$

which exists for all $x \in (a, b)$

$\therefore f(x)$ is derivable in (a, b).

(*c*) Also $f(a) = f(b) = 0$

Thus all the conditions of Rolle's theorem are satisfied. Hence there must exist atleast one value of $c \in (a, b)$ such that $f'(c) = 0$.

Now, $f'(c) = 0 \quad \Rightarrow \quad (c-a)^{m-1}(c-b)^{n-1}[n(c-a) + m(c-b)] = 0$

$\Rightarrow$ either $c - a = 0$ or $c - b = 0$ or $n(c-a) + m(c-b) = 0$

$\Rightarrow$ either $c = a$ or $c = b$ or $c = \dfrac{mb+na}{m+n}$

Now, $c = \dfrac{mb+na}{m+n} \in (a, b)$ $\left[\because \dfrac{mb+na}{m+n} \text{ divides } (a, b) \text{ in the ratio } m:n\right.$

Hence Rolle's theorem is verified.

Example 3. ***Verify Rolle's theorem for $f(x) = \sin 2x$ in $[0, \pi/2]$.***

Solution. Here $f(x) = \sin 2x$

(*a*) Sine function is continuous for all values of x and hence $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$

(*b*) $f'(x) = 2. \cos 2x$ = finite, definite and real for all real values of x

[$\because$ *$\cos\theta$ is real and always lies between* -1 *and* 1]

$\therefore \quad f(x)$ is derivable in $(0, \pi/2)$.

(c) $f(0) = \sin 0 = 0$ and $f(\pi/2) = \sin \pi = 0$

$\Rightarrow \quad f(0) = f(\pi/2)$

Thus all the conditions of Rolle's theorem are satisfied. Hence there must exist atleast one value of $c \in \left(0, \frac{\pi}{2}\right)$ such that $f'(c) = 0$.

We have $\quad f'(x) = 2\cos 2x \quad \Rightarrow \quad f'(c) = 2\cos 2c$

Now, $\quad f'(c) = 0 \quad \Rightarrow \quad 2\cos 2c = 0$

$$\Rightarrow \quad \cos 2c = \cos\frac{\pi}{2} \quad \Rightarrow \quad 2c = \frac{\pi}{2}$$

$$\Rightarrow \quad c = \frac{\pi}{4} \in \left(0, \frac{\pi}{2}\right)$$

Hence Rolle's theorem is verified.

Example 4. ***Verify Rolle's theorem for the function $f(x) = (x^2 - 4x + 3)\, e^{2x}$ in $[1, 3]$*** *[K.U. 2017]*

Solution. Here $f(x) = (x^2 - 4x + 3)\, e^{2x}$

(*i*) The function $x^2 - 4x + 3$, being a polynomial function and e^{2x}, the exponential function are both continuous everywhere. Thus their product $(x^2 - 4x + 3)\, e^{2x}$ is continuous on $[1, 3]$.

(*ii*)
$$\begin{aligned} f'(x) &= (x^2 - 4x + 3)\, e^{2x}.\, 2 + e^{2x}.(2x - 4) \\ &= (2x^2 - 8x + 6 + 2x - 4)\, e^{2x} \\ &= 2\,(x^2 - 3x + 1)\, e^{2x} \end{aligned}$$

$\therefore$ $f'(x)$ exists uniquely on $(1, 3)$ and thus $f(x)$ is derivable on $(1, 3)$.

(*iii*)
$$f(1) = [(1)^2 - 4(1) + 3]\, e^2 = 0.e^2 = 0$$
$$f(3) = [(3)^2 - 4(3) + 3]\, e^6 = 0.e^6 = 0$$

$\therefore \quad f(1) = f(3)$

Thus all the conditions of Rolle's theorem are satisfied.

Hence there must exist atleast one point $x = c$ in $(1, 3)$ such that $f'(c) = 0$

We have $\quad f'(x) = 2(x^2 - 3x + 1)\, e^{2x}$

$\Rightarrow \quad f'(c) = 2\,(c^2 - 3c + 1)\, e^{2c}$

Now $\quad f'(c) = 0 \quad \Rightarrow \quad 2\,(c^2 - 3c + 1)\, e^{2c} = 0 \quad \Rightarrow \quad c^2 - 3c + 1 = 0 \qquad [\because e^{2c} \neq 0]$

$$\therefore \quad c = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

Now, $\quad c = \dfrac{3 + \sqrt{5}}{2} = 2.618 \in (1, 3)$

Hence Rolle's theorem is verified.

Example 5. ***Verify Rolle's theorem for $f(x) = \log(x^2 + 2) - \log 3$ in $[-1, 1]$.***

Solution. Here $f(x) = \log(x^2 + 2) - \log 3$

$$\therefore \quad f'(x) = \frac{1}{x^2+2} \cdot \frac{d}{dx}(x^2+2) - 0 = \frac{2x}{x^2+2}$$

$$= \text{finite, definite and real for all } x \in [-1, 1]$$

$\therefore$ $f(x)$ is derivable for all real values of $x \in (-1, 1)$. But every derivable functio... continuous also. Hence $f(x)$ is continuous for all $x \in [-1, 1]$.

Thus (*i*) $f(x)$ is continuous in $[-1, 1]$

(*ii*) $f(x)$ is derivable in $(-1, 1)$

(*iii*) $f(-1) = \log(1 + 2) - \log 3 = 0$

and $f(1) = \log(1 + 2) - \log 3 = 0$

$\therefore$ $f(-1) = f(1)$

Thus all the conditions of Rolle's theorem are satisfied. Hence there must exist at le... one point $x = c$ in $(-1, 1)$ such that $f'(c) = 0$

We have $f'(x) = \dfrac{2x}{x^2+2} \Rightarrow f'(c) = \dfrac{2c}{c^2+2}$

Now, $f'(c) = 0 \Rightarrow \dfrac{2c}{c^2+2} = 2$

$\Rightarrow c = 0 \in (-1, 1)$.

Hence Rolle's theorem is verified.

Example 6. ***Examine the applicability of Rolle's theorem for the following functio...***

(*i*) $f(x) = (x-1)^{2/5}$ ***on*** $[0, 3]$

(*ii*) $f(x) = \begin{cases} x^2 + 1, & 0 \le x \le 1 \\ 3 - x, & 1 < x \le 2 \end{cases}$ [K.U. 20..]

Solution. (*i*) Here $f(x) = (x - 1)^{2/5}$

$f(x)$ has unique, finite and real values in $[0, 3]$

$\therefore$ $f(x)$ is continuous on the closed interval $[0, 3]$.

$f'(x) = \dfrac{2}{5}(x-1)^{-3/5} = \dfrac{2}{5(x-1)^{3/5}}$, which exists in open interval $(0, 3)$ except at x...

Derivability at $x = 1$:

$$Lf'(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \to 0} \frac{(1-h-1)^{2/5} - 0}{-h}$$

$$= \lim_{h\to 0} \frac{(-h)^{2/5}}{-h} = \lim_{h\to 0} \frac{1}{(-h)^{3/5}}$$

which does not exist.

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h-1)^{2/5}-0}{h}$$

$$= \lim_{h\to 0} \frac{h^{2/5}}{h} = \lim_{h\to 0} \frac{1}{h^{3/5}}$$

which does not exist.

$\therefore f(x)$ is not derivable at $x = 1 \in (0, 3)$.

Thus $f(x)$ is not derivable in the open interval $(0, 3)$ and hence Rolle's theorem is not applicable.

(*ii*) We have $f(x) = \begin{cases} x^2+1, & 0 \le x \le 1 \\ 3-x, & 1 < x \le 2 \end{cases}$

Here $f(1) = 2$

The given function being a polynomial function is continuous and derivable everywhere except possible at $x = 1$.

Derivability at $x = 1$:

$$Rf'(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0} \frac{[3-(1+h)]-2}{h} = \lim_{h\to 0} -\frac{h}{h} = -1$$

$$Lf'(1) = \lim_{h\to 0} \frac{f(1-h)-f(1)}{-h}$$

$$= \lim_{h\to 0} \frac{[(1-h)^2+1]-2}{-h} = \lim_{h\to 0} \frac{1+h^2-2h+1-2}{-h}$$

$$= \lim_{h\to 0} \frac{h^2-2h}{-h} = \lim_{h\to 0} (-h+2) = 2$$

Thus we have $Lf'(1) \neq Rf'(1)$

$\therefore f(x)$ is not derivable at $x = 1$.

Hence Rolle's theorem is not applicable.

Example 7. ***If $f(x)$ is a polynomial, then show that between any two consecutive roots of $f'(x) = 0$, there lies at the most one root of $f(x) = 0$.***

Solution. Let a, b be two consecutive roots of $f'(x) = 0$ such that $a < b$.

Suppose that $f(x) = 0$ has two different roots α, β such that α, β lies between a and b

$$\therefore \qquad f(\alpha) = f(\beta) = 0$$

Also $f(x)$ being a polynomial is continuous on $[\alpha, \beta]$ and derivable on (α, β).

Thus $f(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ There exists atleast one $c \in (\alpha, \beta)$ such that $f'(c) = 0$

$\Rightarrow$ c is a root of $f'(x) = 0$

$\therefore$ Between two consecutive roots a and b of $f'(x) = 0$, there lies a root of $f'(x) = 0$

[$\because$ *c lies between a and b*]

which is a contradiction. Hence our supposition is wrong.

$\therefore$ Between any two consecutive roots of $f'(x) = 0$, there lies atmost one root of $f(x)$.

Example 8. ***Show that between any two roots of $e^x \cos x - 1 = 0$, there exists atleast one root of $e^x \sin x - 1 = 0$.***

[*M.D.U. 2017*]

Solution. The given equation is $e^x \cos x - 1 = 0$

Let α and β be two roots of equation (1)

$$\therefore \qquad e^\alpha \cos \alpha - 1 = 0 \quad \text{and} \quad e^\beta \cos \beta - 1 = 0$$

Define a function $f(x) = e^{-x} - \cos x$ for all $x \in [\alpha, \beta]$

$$f(\alpha) = e^{-\alpha} - \cos \alpha = \frac{1}{e^\alpha} - \cos \alpha = \frac{1 - e^\alpha \cos \alpha}{e^\alpha}$$

$$f(\beta) = e^{-\beta} - \cos \beta = \frac{1}{e^\beta} - \cos \beta = \frac{1 - e^\beta \cos \beta}{e^\beta}$$

Using (2), we get $f(\alpha) = 0$ and $f(\beta) = 0$

To apply Rolle's theorem on $f(x)$:

(*i*) f is continuous function in the closed interval $[\alpha, \beta]$

(*ii*) $f'(x) = -e^{-x} + \sin x$, which exists for all $x \in (\alpha, \beta)$

$\therefore$ f is derivable in open interval (α, β)

(*iii*) $f(\alpha) = f(\beta) = 0$ [*Using* (3)]

Thus f satisfy all the conditions of Rolle's theorem.

$\therefore$ There exists some point $c \in (\alpha, \beta)$ such that $f'(c) = 0$

$$\Rightarrow \qquad -e^{-c} + \sin c = 0$$

$$\Rightarrow \qquad \sin c - \frac{1}{e^c} = 0$$

$$\Rightarrow \qquad e^c \cdot \sin c - 1 = 0$$

$\Rightarrow$ c is a root of equation $e^x \sin x - 1 = 0$, where $\alpha < c < \beta$.

Hence the result.

Example 9. ***Prove that if $a_0, a_1, \ldots, a_n$ are real numbers such that***

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0,$$

then there exists atleast one real number x between 0 and 1 such that

$$a_0 x^n + a_1 x^{n-1} + \ldots + a_n = 0.$$ [K.U. 2016]

Solution. Consider $f(x) = \frac{a_0}{n+1} x^{n+1} + \frac{a_1}{n} x^n + \ldots + \frac{a_{n-1}}{2} x^2 + a_n x, \; x \in [0, 1]$

$\therefore$ $f(0) = 0$

$$f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \ldots + \frac{a_{n-1}}{2} + a_n = 0$$ [Given]

$f(x)$ being a polynomial is continuous on closed inteval $[0, 1]$ and derivable on open interval $(0, 1)$.

Also $f(0) = f(1) = 0$

Thus $f(x)$ satisfy all the conditions of Rolle's theorem.

$\therefore$ There exists atleast one $x \in (0, 1)$ such that $f'(x) = 0$

or $$\frac{a_0}{n+1}(n+1)x^n + \frac{a_1}{n} \cdot nx^{n-1} + \ldots + \frac{a_{n-1}}{2}(2x) + a_n = 0$$

or $$a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$$

Hence the result.

Example 10. ***Show that there is no real number k for which the equation $x^3 - 3x + k = 0$ has two distinct roots in $[0, 1]$.*** [K.U. 2014, 12]

Solution. Let $f(x) = x^3 - 3x + k$ ...(1)

Suppose that there are two real roots of $f(x) = 0$ lying in $[0, 1]$

Let α, β be two such roots

$\therefore$ $f(\alpha) = 0 = f(\beta)$ and $[\alpha, \beta] \subseteq [0, 1]$

To apply Rolle's theorem on $f(x)$ in $[\alpha, \beta]$.

(i) $f(x)$ being a polynomial is continuous in $[\alpha, \beta]$

(ii) $f'(x) = 3x^2 - 3$ which exists in (α, β)

$\therefore$ f is derivable in (α, β)

(iii) $f(\alpha)=f(\beta)=0$

Thus $f(x)$ satisfy all the conditions of Rolle's theorem.

$\therefore$ There exists $c \in (\alpha, \beta)$ such that $f'(c)=0$

$\Rightarrow \quad 3c^2-3=0 \Rightarrow c^2=1 \Rightarrow c=\pm 1,$

which does not lie in (α, β). This is a contradiction.

Hence our supposition is wrong and the result follows.

EXERCISE 2.2

Verify Rolle's theorem for the following functions in the given intervals [Q.

1. (i) $f(x)=x^3-6x^2+11x-6$ in $[1, 3]$

   (ii) $f(x)=x^3+3x^2-24x-80$ in $[-4, 5]$

   (iii) $f(x)=x^3-9x^2+26x-24$ in $[2, 4]$

2. (i) $f(x)=\dfrac{x^2-3x-4}{x-5}$ in $[-1, 4]$

   (ii) $f(x)=\sqrt{4-x^2}$ in $[-2, 2]$

   (iii) $f(x)=|x^2-4|$ in $[-1, 1]$

3. (i) $f(x)=\cos 2x$ in $\left[-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right]$

   (ii) $f(x)=\cos 2\left(x-\dfrac{\pi}{4}\right)$ in $\left[0, \dfrac{\pi}{2}\right]$

   (iii) $f(x)=\sin x-\sin 2x$ in $[0, \pi]$

4. (i) $f(x)=\log\left[\dfrac{x^2+ab}{(a+b)x}\right]$ in $[a, b]$, where a and b are positive numbers.

   (ii) $f(x)=e^{1-x^2}$ in $[-1, 1]$ (iii) $f(x)=e^x \cos x$ in $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

5. (i) $f(x)=\dfrac{\sin x}{e^x}$ in $[0, \pi]$ (ii) $f(x)=x(x+3)\,e^{-x/2}$ in [-3,

   (iii) $f(x)=\tan x$ in $[0, \pi]$

6. Examine the applicability of Rolle's theorem for the following functions :

   (i) $f(x)=|x|$ in $[-1, 1]$ (ii) $f(x)=x^{2/3}$ in $[-1, 1]$

   (iii) $f(x)=2+(x-1)^{2/3}$ in $[0, 2]$ (iv) $f(x)=\begin{cases}-4x+5, & 0\le x\le 1\\ 2x-3, & 1<x\le 2\end{cases}$

   [K.U. 2005]

2.9. LAGRANGE'S MEAN VALUE THEOREM

Statement. ***If a function $f(x)$ is defined in the closed interval $[a, b]$ such that***

(i) $f(x)$ is continuous in closed interval $[a, b]$

(ii) $f(x)$ is derivable in open interval (a, b), then there exists atleast one poi

$c \in (a, b)$ such that $\dfrac{f(b)-f(a)}{b-a} = f'(c)$.

[M.D.U. 2018, 17, 12, 10, 06; K.U. 2

Proof. Let us consider a function $F(x) = f(x) + Ax$ ...(1)

where A is a constant to be determined such that

$$F(a) = F(b) \qquad \ldots(2)$$

Now, $F(a) = f(a) + Aa$ and $F(b) = f(b) + Ab$

Using (2), we have $\quad f(a) + Aa = f(b) + Ab$

$\therefore \quad f(b) - f(a) = -A(b-a)$

$$\Rightarrow \qquad -A = \frac{f(b)-f(a)}{b-a} \qquad \ldots(3)$$

Now, $f(x)$ is given to be continuous in $[a, b]$ and Ax being a polynomial in x, is alw continuous in $[a, b]$. Thus $F(x)$, the sum of two continuous functions $f(x)$ and Ax is also continu in $[a, b]$ and also $F(x)$, the sum of two derivable functions $f(x)$ and Ax in (a, b) is also derivable (a, b).

Thus we conclude that $F(x) = f(x) + Ax$ is

(i) continuous in the closed interval $[a, b]$ (*i.e.*, $a \le x \le b$)

(ii) derivable in the open interval (a, b) (*i.e.*, $a < x < b$)

and (iii) $F(a) = F(b)$ [Assum

$\therefore$ $F(x)$ satisfies all the three conditions of Rolle's theorem and hence there must ex atleast one value 'c' of x in open interval (a, b) such that $F'(c) = 0$

Now, $\quad F(x) = f(x) + Ax$

$\therefore \quad F'(x) = f'(x) + A \Rightarrow F'(c) = f'(c) + A$

Now $\quad F'(c) = 0 \Rightarrow f'(c) + A = 0 \Rightarrow f'(c) = -A \qquad \ldots(4)$

From (3) and (4), we have

$$\frac{\boldsymbol{f(b)-f(a)}}{\boldsymbol{b-a}} = \boldsymbol{f'(c)} \text{ for some } c \in (a, b).$$

SOLVED EXAMPLES

Example 1. ***Verify Lagrange's mean value theorem for***

(*i*) $f(x) = 2x - x^2$ *in* $[0, 1]$ (*ii*) $f(x) = x(x+4)^2$ *in* $[0, 4]$

Solution. (*i*) Here $f(x) = 2x - x^2$

$f(x)$ being a polynomial function of x is continuous for all real x.

$\therefore$ $f(x)$ is continuous in $[0, 1]$

$f'(x) = 2 - 2x$ which exists for all $x \in (0, 1)$. Thus $f(x)$ is derivable in $(0, 1)$.

Thus both the conditions of Lagrange's mean value theorem are satisfied. Hence there must exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ where } a = 0,\ b = 1. \quad ...(1)$$

Now, $f'(x) = 2 - 2x \Rightarrow f'(c) = 2 - 2c$

$f(a) = f(0) = 0 - 0 = 0$

and $f(b) = f(1) = 2 - 1 = 1$

$\therefore$ From (1), $\dfrac{1-0}{1-0} = 2 - 2c$

$\Rightarrow$ $2 - 2c = 1 \Rightarrow 2c = 1 \Rightarrow c = \dfrac{1}{2} \in (0, 1)$

Hence, Lagrange's mean value theorem is verified.

(*ii*) Here $f(x) = x(x+4)^2 = x^3 + 8x^2 + 16x$

$f(x)$ being a polynomial function is continuous in $[0, 4]$

$f'(x) = 3x^2 + 16x + 16$ which exists for all $x \in (0, 4)$. Thus $f(x)$ is derivable in $(0, 4)$.

Thus both the conditions of Lagrange's mean value theorem are satisfied. Hence there must exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ where } a = 0,\ b = 4 \quad ...(1)$$

Now, $f'(x) = 3x^2 + 16x + 16$

$\Rightarrow$ $f'(c) = 3c^2 + 16c + 16$

and $f(b) = f(4) = 4(4+4)^2 = 256$

$f(a) = f(0) = 0(0+4)^2 = 0$

$\therefore$ From (1), $\dfrac{256-0}{4-0} = 3c^2 + 16c + 16$

$\Rightarrow \quad 3c^2 + 16c + 16 = 64 \quad \Rightarrow \quad 3c^2 + 16c - 48 = 0$

$$\Rightarrow \quad c = \frac{-8 \pm 4\sqrt{13}}{3}$$

Now, $$c = \frac{-8 + 4\sqrt{13}}{3} = 2.13 \in (0, 4)$$

Hence, Lagrange's mean value theorem is verified.

Example 2. ***Find 'c' of the Lagranges mean value theorem if $f(x) = (x-1)(x-2)(x-3)$ and $a = 0, b = 4$.***

Solution. Here $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$

$f(x)$ being a polynomial in x is continuous in $[0, 4]$.

$f'(x) = 3x^2 - 12x + 11$ which exists for all $x \in (0, 4)$. Thus $f(x)$ is derivable in $(0, 4)$

Thus both the condition of Lagrange's mean value theorem are satisfied. Hence there must exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = f'(c) \text{ where } a = 0, \quad b = 4 \qquad ...(1)$$

Now $$f'(x) = 3x^2 - 12x + 11$$

$$\Rightarrow \quad f'(c) = 3c^2 - 12c + 11$$

$$f(b) = f(4) = (4-1)(4-2)(4-3) = 3.2.1. = 6$$

and $$f(a) = f(0) = (-1)(-2)(-3) = -6$$

$\therefore$ From (1), $\dfrac{6-(-6)}{4-0} = 3c^2 - 12c + 11$

$\Rightarrow \quad 3c^2 - 12c + 11 = 3 \quad \Rightarrow \quad 3c^2 - 12c + 8 = 0$

$$\therefore \quad c = \frac{12 \pm \sqrt{144 - 96}}{6} = \frac{6 \pm 2\sqrt{3}}{3}$$

Both the above value of c lie between 0 and 4.

Example 3. ***Examine the validity of Lagrange's mean value theorem for the function $f(x) = x^{1/3}$ in the interval $[-1, 1]$.***

Solution. Here $f(x) = x^{1/3}$

(a) $f(x)$ is continuous in $[-1, 1]$

(b) $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$, which exists for all $x \in (-1, 1)$ except at $x = 0$.

Derivability at $x = 0$:

$$Rf'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{h^{1/3} - 0}{h} = \lim_{h \to 0} \frac{1}{h^{2/3}}$$

which does not exist.

Thus $f'(x)$ does not exist for $x = 0 \in (-1, 1)$

$\Rightarrow$ $f(x)$ is not derivable in $(-1, 1)$

Hence, Lagrange's mean value theorem is not applicable.

Example 4. ***Verify Lagranges mean value theorem for $f(x) = \sin x$ in $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.***

[K.U. 2014]

Solution. Here $f(x) = \sin x$

Now, $\sin x$ is finite, definite and real for all $x \in \left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$

$\therefore$ $f(x)$ is continuous in $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$

$f'(x) = \cos x$, which is finite, definite and real for all $x \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$

$\therefore$ $f(x)$ is derivable in $\left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$

Thus both the conditions of Lagrange's mean value theorem are satisfied. Hence there must exist atleast one point $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ where } a = \frac{\pi}{2},\ b = \frac{5\pi}{2} \qquad ...(1)$$

Now,

$$f'(x) = \cos x \Rightarrow f'(c) = \cos c$$

$$f(b) = f\left(\frac{5\pi}{2}\right) = \sin\frac{5\pi}{2} = 1$$

and

$$f(a) = f\left(\frac{\pi}{2}\right) = \sin\frac{\pi}{2} = 1$$

$\therefore$ From (1), $$\frac{1-1}{\frac{5\pi}{2} - \frac{\pi}{2}} = \cos c \Rightarrow \cos c = 0$$

$$\Rightarrow \qquad c = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

Now, $$c = \frac{3\pi}{2} \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right)$$

Hence Lagrange's mean value theorem is verified.

Example 5. ***Verify the conditions of mean value theorem for the function $f(x) = \log$ on [1, 2]. Find a point c in the interval stated by mean value theorem.***

Solution. Here $f(x) = \log x$

$f(x)$ is continuous in [1, 2] as $\log x$ is finite, definite and real for all $x > 0$.

$f'(x) = \frac{1}{x}$ which exists for all $x \neq 0$. Thus $f(x)$ is derivable in (1, 2)

Thus both the conditions of Lagrange's mean value theorem are satisfied. Hence the must exist atleast one $c \in (1, 2)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) \text{ where } a = 1,\ b = 2$$

Now, $$f'(x) = \frac{1}{x} \Rightarrow f'(c) = \frac{1}{c}$$

$$f(b) = f(2) = \log 2 \quad \text{and} \quad f(a) = f(1) = \log 1$$

$\therefore$ From (1), $$\frac{\log 2 - \log 1}{2 - 1} = \frac{1}{c}$$

$$\Rightarrow \qquad \log 2 - \log 1 = \frac{1}{c} \Rightarrow \log 2 = \frac{1}{c} \qquad [\because \log 1 =$$

$$\therefore \qquad c = \frac{1}{\log 2} = \log_2 e.$$

Example 6. ***Using Lagrange's mean value theorem, prove that***

$$|\sin m - \sin n| \leq |m - n| \text{ for all } m, n.$$

Solution. Let $f(x) = \sin x$

$$\therefore \qquad f'(x) = \cos x$$

Now, $f(x)$ is continuous in $[n, m]$ and $f'(x)$ exists in (n, m), $n < m$.

Thus both the conditions of Lagrange's mean value theorem are satisfied, therefore there must exist atleast one point $c \in (m, n)$ such that

$$\frac{\sin m - \sin n}{m - n} = f'(c) = \cos c$$

$$\Rightarrow \quad \left| \frac{\sin m - \sin n}{m - n} \right| = |\cos c| \leq 1$$

$$\Rightarrow \quad |\sin m - \sin n| \leq |m - n|$$

Example 7. ***Show that*** $x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}, \; x > 0.$ [K.U. 2018, 15, 06]

Solution. Let $f(x) = \log(1+x) - \left(x - \dfrac{x^2}{2}\right)$

Differentiating w.r.t. x, we have

$$f'(x) = \frac{1}{1+x} - (1-x)$$

$$= \frac{1-(1-x^2)}{1+x} = \frac{x^2}{1+x} > 0 \qquad [\because x > 0]$$

$\therefore f(x)$ is an increasing function of x for all $x > 0$

$$\therefore \quad f(x) > f(0) \text{ for all } x > 0$$

$$\Rightarrow \quad \log(1+x) - \left(x - \frac{x^2}{2}\right) > 0 \qquad [\because f(0) = 0]$$

$$\Rightarrow \quad \log(1+x) > x - \frac{x^2}{2}$$

$$\Rightarrow \quad x - \frac{x^2}{2} < \log(1+x), \; x > 0 \qquad \text{...(1)}$$

Let $\quad g(x) = \left[x - \dfrac{x^2}{2(1+x)}\right] - \log(1+x)$

Differentiating w.r.t. x, we have

$$g'(x) = 1 - \frac{1}{2}\left[\frac{2x + x^2}{(1+x)^2}\right] - \frac{1}{1+x}$$

$$= \frac{2(1+x)^2 - (2x+x)^2 - 2(1+x)}{2(1+x)^2}$$

$$= \frac{(2+2x^2+4x) - (2x+x^2) - (2+2x)}{2(1+x)^2}$$

$$= \frac{x^2}{2(1+x)^2} > 0$$

$\therefore$ $g(x)$ is an increasing function of x for all $x > 0$

$$\Rightarrow \qquad g(x) > g(0) \text{ for all } x > 0$$

$$\Rightarrow \qquad x - \frac{x^2}{2(1+x)} - \log(1+x) > 0 \qquad [\because g(0) = \ldots$$

$$\Rightarrow \qquad x - \frac{x^2}{2(1+x)} > \log(1+x)$$

$$\Rightarrow \qquad \log(1+x) < x - \frac{x^2}{2(1+x)} \text{ for all } x > 0 \qquad \ldots$$

From (1) and (2), $\quad x - \dfrac{x^2}{2} < \log(1+x) < x - \dfrac{x^2}{2(1+x)}, \quad x > 0.$

Example 8. ***Use mean value theorem to prove that***

$1 + x < e^x < 1 + xe^x$ for all $x > 0$. *[K.U. 2007; M.D.U. 200...*

Solution. Consider $f(x) = e^x$; $[0, x]$

Here f is continuous on $[0, x]$ and derivable on $(0, x)$, therefore by mean value theorem there exists some $c \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c)$$

or $$\frac{e^x - e^0}{x - 0} = e^c$$

or $$\frac{e^x - 1}{x} = e^c \qquad \ldots($$

As $\quad 0 < c < x$

$\therefore \quad e^0 < e^c < e^x$

or $\quad 1 < e^c < e^x$

or $\quad 1 < \dfrac{e^x - 1}{x} < e^x$ $\qquad$ *[Using (1*

2.11. CAUCHY'S MEAN VALUE THEOREM

Let f and g be two functions defined on closed interval $[a, b]$ such that

(i) f and g are continuous on closed interval $[a, b]$

(ii) f and g are derivable on open interval (a, b)

(iii) $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists atleast one point $c \in (a, b)$

such that $$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

[*M.D.U. 2007*]

Proof. The given conditions imply that $g(a) \neq g(b)$

Suppose that $g(a) = g(b)$. In this case g satisfy all the conditions of Rolle's theorem. So there exists atleast one point $c \in (a, b)$ such that $g'(c) = 0$.

This is a contradiction to the given fact that $g'(x) \neq 0$ for all $x \in (a, b)$.

Thus our supposition that $g(a) = g(b)$ is wrong and hence $g(a) \neq g(b)$.

Let $\phi(x) = f(x) + A\,g(x)$ where A is constant to be determined such that $\phi(a) = \phi(b)$

$$\Rightarrow \quad f(a) + A\,g(a) = f(b) + A\,g(b)$$

$$\Rightarrow \quad A\,[g(a) - g(b)] = f(b) - f(a)$$

$$\Rightarrow \quad A = \frac{f(b)-f(a)}{g(a)-g(b)}$$

$$\Rightarrow \quad -A = \frac{f(b)-f(a)}{g(b)-g(a)} \qquad \text{...(1)}$$

To apply Rolle's theorem on $\phi(x)$:

(*i*) f and g are continuous functions on $[a, b]$ [*Given*]

$\therefore$ $\phi(x) = f(x) + A\,g(x)$ is a continuous function on $[a, b]$.

(*ii*) f and g are derivable in open interval (a, b) [*Given*]

$\therefore$ $\phi(x) = f(x) + A\,g(x)$ is derivable function in (a, b).

(*iii*) $\phi(a) = \phi(b)$ (assumed).

Thus $\phi(x)$ satisfy all the conditions of Rolle's theorem.

$\therefore$ There exists atleast one point $c \in (a, b)$ such that $\phi'(c) = 0$

$$\Rightarrow \quad f'(c) + A\,g'(c) = 0$$

$$\Rightarrow \qquad \frac{f'(c)}{g'(c)} = -A$$

$$\Rightarrow \qquad \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{\ \ )-g(a)}.$$ [Using

Remark.

Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem.

If $g(x) = x$, then $g'(x) = 1, g(b) = b$ and $g(a) = a$.

Putting these values in Cauchy's mean value theorem, we get

$$\frac{f'(c)}{1} = \frac{f(b)-f(a)}{b-a}$$

which is Lagrange's mean value theorem.

SOLVED EXAMPLES

Example 1. ***Verify the Cauchy's mean value theorem for $f(x) = x^2$, $g(x) = x^3$ in [1,***

Solution. Here $f(x) = x^2$ and $g(x) = x^3$

(*i*) f and g being polynomial functions are continuous in [1, 2]

(*ii*) f and g being polynomial functions are derivable in (1, 2)

(*iii*) $g'(x) = 3x^2 \neq 0$ for all $x \in (1, 2)$.

Thus the conditions of Cauchy's mean value theorem are satisfied. So there exists so point $c \in (1, 2)$ such that

$$\frac{f(2)-f(1)}{g(2)-g(1)} = \frac{f'(c)}{g'(c)}$$

or $$\frac{4-1}{8-1} = \frac{2c}{3c^2}$$

or $$\frac{3}{7} = \frac{2}{3c}$$

or $$c = \frac{14}{9} \text{ which lies in } (1, 2)$$

Hence, Cauchy's mean value theorem is verified.

Example 2. ***Show that*** $\dfrac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta,\ 0 < \alpha < \theta < \beta < \dfrac{\pi}{2}$.

Solution. Consider two functions f and g such that

$$f(x) = \sin x \quad\text{and}\quad g(x) = \cos x \text{ for all } x \in [\alpha, \beta],\ 0 < \alpha < \beta < \frac{\pi}{2}$$

To apply Cauchy's mean value theorem on $f(x)$ and $g(x)$:

(*i*) $f(x) = \sin x$ and $g(x) = \cos x$ are continuous functions in the closed interval $[\alpha, \beta]$.

(*ii*) $f'(x) = \cos x$ and $g'(x) = -\sin x$

Function f and g are derivable in (α, β)

(*iii*) $g'(x) = -\sin x \neq 0$ for all $x \in (\alpha, \beta)$.

Thus the conditions of Cauchy's mean value theorem are satisfied and so there exists some point $\theta \in (\alpha, \beta)$ such that

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \quad \frac{\sin\beta - \sin\alpha}{\cos\beta - \cos\alpha} = \frac{\cos\theta}{-\sin\theta}$$

$$\Rightarrow \quad -\frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = -\cot\theta$$

$$\Rightarrow \quad \frac{\sin\alpha - \sin\beta}{\cos\beta - \cos\alpha} = \cot\theta \text{ where } 0 < \alpha < \theta < \beta < \frac{\pi}{2}.$$

EXERCISE 2.4

Verify Cauchy's mean value theorem for the following functions:

1. (*i*) $f(x) = \sin x,\ g(x) = \cos x$ on $\left[-\frac{\pi}{2}, 0\right]$

(*ii*) $f(x) = e^x,\ g(x) = e^{-x}$ on $[0, 1]$

2. Let the function f be continuous in $[a, b]$ and derivable in (a, b). Show that there exists a number c in (a, b) such that $2c\,[f(a) - f(b)] = f'(c)\,[a^2 - b^2]$.

3. If f' and g' exist for all $x \in [a, b]$ and if $g'(x) \neq 0$ for all $x \in (a, b)$, then prove that for some $c \in (a, b)$ $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$.

4. If f' and g' are continuous and differentiable on $[a, b]$, then show that for $a < c < b$

$$\frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \frac{f''(c)}{g''(c)}.$$

ANSWERS

1. (i) $c = -\frac{\pi}{4}$ (ii) $c = \frac{1}{2}$.

2.12. TAYLOR'S THEOREM

If a function f defined on $[a, a+h]$ is such that

(i) $f, f', f'', \ldots, f^{n-1}$ are continuous functions of x on $[a, a+h]$

(ii) $f^n(x)$ exists on $(a, a+h)$, then there exists atleast one real numb θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \ldots$$
$$\ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{h^n(1-\theta)^{n-p}}{p(n-1)!}f^n(a+\theta h)$$

Proof. Consider the function

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots$$
$$\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + (a+h-x)^p.A$$

where A is constant to be determined such that $\phi(a) = \phi(a+h)$

Putting $x = a$, $a+h$ in (1), we get

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + h^p.A$$

$$\phi(a+h) = f(a+h)$$

Since $\phi(a+h) = \phi(a)$

$$\therefore \quad f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + h^p.A$$

To apply Rolle's theorem on $f(x)$:

(i) $f(x), f'(x), f''(x), \ldots, f^{n-1}(x)$ are continuous on $[a, a+h]$ [Gi...

$(a+h-x), (a+h-x)^2, \ldots, (a+h-x)^{n-1}, (a+h-x)^p$, being polynomials are continuous on the closed interval $[a, a+h]$

Example 3. ***Show that*** $\log(x+h) = \log x + \dfrac{h}{x} - \dfrac{h^2}{2x^2} + \dots + (-1)^{n-1}\dfrac{h^n}{n(x+\theta h)^n}$. ...(A)

Solution. Let
$$f(x+h) = \log(x+h)$$

$\therefore$
$$f(x) = \log x$$

$\therefore$
$$f'(x) = \frac{1}{x}$$

$$f''(x) = -\frac{1}{x^2}$$

$$f'''(x) = \frac{(-1)(-2)}{x^3} = (-1)^2\frac{2!}{x^3}$$

$$f^{iv}(x) = \frac{(-1)(-2)(-3)}{x^4} = (-1)^3\frac{3!}{x^4}$$

....

....

$$f^{n-1}(x) = \frac{(-1)^{n-2}(n-2)!}{x^{n-1}}$$

$$f^n(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$$

$\therefore$
$$f^n(x+\theta h) = (-1)^{n-1}\frac{(n-1)!}{(x+\theta h)^n}$$

By Taylor's theorem with Lagrange's form of remainder, we have

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(x) + \frac{h^n}{n!}f^n(x+\theta h)$$

Substituting the values from (A) in above equation, we have

$$\log (x+h) = \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3!}\frac{2!}{x^3} + \ldots.$$

$$\ldots + \frac{h^{n-1}}{(n-1)!}(-1)^{n-2}\frac{(n-2)!}{x^{n-1}} + \frac{h^n}{n!}(-1)^{n-1}\frac{(n-1)!}{(x+\theta h)^n}$$

$$= \log x + \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} + \ldots + (-1)^{n-2}\frac{h^{n-1}}{(n-1)x^{n-1}} + (-1)^{n-1}\frac{h^n}{n(x+\theta h)^n}$$

Example 4. ***Show that for every value of x,***

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!}\sin \theta x, \quad 0<\theta<1.$$

[K.U. ...]

Solution. Let $f(x) = \sin x$

We know that $f^n(x) = \sin\left[x + \frac{n\pi}{2}\right]$

$$f^{2n-1}(x) = \sin\left[x + (2n-1)\frac{\pi}{2}\right]$$

$$f^{2n}(x) = \sin\left[x + 2n\cdot\frac{\pi}{2}\right]$$

Hence,

$$\left.\begin{aligned}
f(0) &= \sin 0 = 0\\
f'(0) &= \sin\left(\frac{\pi}{2}\right) = 1\\
f''(0) &= \sin\left(\frac{2\pi}{2}\right) = \sin \pi = 0\\
f'''(0) &= \sin\left(\frac{3\pi}{2}\right) = -1\\
f^{iv}(0) &= \sin\left(\frac{4\pi}{2}\right) = \sin 2\pi = 0\\
f^{v}(0) &= \sin \frac{5\pi}{2} = 1\\
f^{2n-1}(0) &= \sin\left[(2n-1)\frac{\pi}{2}\right]\\
&= \sin\left(n\pi - \frac{\pi}{2}\right) = (-1)^n \sin\left(-\frac{\pi}{2}\right)\\
&= (-1)^{n+1} = (-1)^{n-1}\\
f^{2n}(\theta x) &= \sin(\theta x + n\pi) = (-1)^n \sin \theta x
\end{aligned}\right\}$$

By Maclaurin's theorem, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \frac{x^5}{5!} f^{v}(0) + \ldots$$

$$\ldots + \frac{x^{2n-1}}{(2n-1)!} f^{2n-1}(0) + \frac{x^{2n}}{(2n)!} f^{2n}(\theta x)$$

Substituting the values from (A) in above equation, we have

$$\sin x = 0 + x \cdot 1 + \frac{x^2}{2!} \cdot 0 + \frac{x^3}{3!}(-1)^1 + \frac{x^4}{4!} \cdot 0 + \frac{x^5}{5!}(-1)^2 + \ldots$$

$$\ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + \frac{x^{2n}}{(2n)!}(-1)^n \sin \theta x$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!} \sin \theta x .$$

xample 5. ***Expand $e^{ax} \sin bx$ by Maclaurin's theorem with Cauchy's form of remainder er n terms.***

Solution. Let $f(x) = e^{ax} \sin bx$

$$\therefore \quad f'(x) = ae^{ax} \sin bx + e^{ax} b \cos bx$$

$$= e^{ax}(a \sin bx + b \cos bx)$$

$$f''(x) = ae^{ax}(a \sin bx + b \cos bx) + e^{ax}(ab \cos bx - b^2 \sin bx)$$

$$= e^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx]$$

$$f'''(x) = ae^{ax}[(a^2 - b^2) \sin bx + 2ab \cos bx] + e^{ax}[b(a^2 - b^2) \cos bx - 2ab^2 \sin bx]$$

$$= e^{ax}[(a^3 - 3ab^2) \sin bx + b(3a^2 - b^2) \cos bx]$$

....

....

We know that $f^n(x) = (a^2 + b^2)^{n/2} e^{ax} \sin\left(bx + n \tan^{-1} \frac{b}{a}\right)$

Let $x = 0$, $\therefore$ $f(0) = 0$,

$$f'(0) = b$$

$$f''(0) = 2ab$$

$$f'''(0) = b(3a^2 - b^2)$$

$$f^n(\theta x) = (a^2 + b^2)^{n/2} e^{a\theta x} \sin\left(b\theta x + n \tan^{-1} \frac{b}{a}\right)$$

Putting these values in Maclaurin's theorem with Cauchy's remainder, we have

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x) \text{ where } 0<\theta<1$$

$$\therefore \quad e^{ax}\sin bx=0+x\,.\,b+\frac{x^2}{2!}2ab+\frac{x^3}{3!}b(3a^2-b^2)+\ldots$$

$$\ldots+\frac{x^n}{(n-1)!}(1-\theta)^{n-1}(a^2+b^2)^{n/2}e^{a\theta x}\sin\left(b\theta x+n\tan^{-1}\ldots\right)$$

$$=bx+2ab\,.\,\frac{x^2}{2!}+b(3a^2-b^2)\frac{x^3}{3!}+\ldots$$

$$\ldots+(a^2+b^2)^{n/2}(1-\theta)^{n-1}\frac{x^n}{(n-1)!}e^{a\theta x}\sin\left(b\theta x+n\tan^{-1}\ldots\right)$$

Example 6. ***Expand $\sqrt{x}$ in ascending powers of x by Maclaurin's theorem, if possi…***

Solution. Let $f(x)=\sqrt{x}$

$$\therefore \quad f'(x)=\frac{1}{2\sqrt{x}} \quad \text{and} \quad f''(x)=-\frac{1}{4x^{3/2}}$$

Putting $x=0$, we get

$$f'(0)=\infty \quad \text{and} \quad f''(0)=\infty$$

This shows that $f(x)=\sqrt{x}$ cannot be expanded in ascending powers of x by Maclauri… theorem.

Example 7. ***The expansion of function $f(x)=(1-x)^{5/2}$ is***

$$f(0)+xf'(0)+\frac{x^2}{2}f''(\theta x).$$ [K.U. 20…

Find the value of θ as $x\to 1$.

Solution. Here $f(x)=(1-x)^{5/2} \Rightarrow f(0)=1$

$$\therefore \quad f'(x)=-\frac{5}{2}(1-x)^{3/2} \Rightarrow f'(0)=-\frac{5}{2}$$

…nd

$$f''(x)=\frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta x)=\frac{15}{4}(1-\theta x)^{1/2}$$

Now,

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2}f''(\theta x)$$ [Gi…

$$\therefore \quad (1-x)^{5/2} = 1 - \frac{5}{2}x + \frac{15}{8}[1-\theta x]^{1/2} x^2$$

When $x \to 1$, we have

$$0 = 1 - \frac{5}{2} + \frac{15}{8}(1-\theta)^{1/2}$$

or $$\frac{15}{8}(1-\theta)^{1/2} = \frac{3}{2}$$

or $$(1-\theta)^{1/2} = \frac{3}{2} \times \frac{8}{15}$$

or $$(1-\theta)^{1/2} = \frac{4}{5} \quad \Rightarrow \quad 1-\theta = \frac{16}{25} \quad \Rightarrow \quad \theta = \frac{9}{25}.$$

EXERCISE 2.5

1. State and prove Taylor's theorem with Lagrange's form of remainder after n terms.

2. State and prove Maclaurin's theorem with Lagrange's form of remainder after n terms.

3. (a) State and prove Taylor's development of a function with Cauchy's form of remainder after n terms and hence deduce the Maclaurin's expansions.

   (b) State and prove Maclaurin's theorem with Cauchy's form of remainder.

4. Expand a^x by Maclaurin's theorem with Lagrange's form of remainder after n terms.

5. Show by means of Maclaurin's expansion that

   (i) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \ldots\ldots + (-1)^n \dfrac{x^{2n}}{2n!} + (-1)^{n+1} \dfrac{x^{2n+1}}{(2n+1)!} \sin(\theta x)$

   (ii) $\log(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots\ldots + (-1)^{n-2} \dfrac{x^{n-1}}{(n-1)} + (-1)^{n-1} \dfrac{x^n}{n(1+\theta x)^n}$

   (iii) $\log(1-x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \ldots\ldots - \dfrac{x^{n-1}}{n-1} - \dfrac{x^n}{n(1-\theta x)^n}$

   (iv) $e^{ax}\cos bx = 1 + ax + (a^2-b^2)\dfrac{x^2}{2!} + a(a^2-3b^2)\dfrac{x^3}{3!} + \ldots\ldots$

   $\ldots\ldots + \dfrac{x^n}{n!}(a^2+b^2)^{n/2} e^{a\theta x} \cos\left(b\theta x + n\tan^{-1}\dfrac{b}{a}\right)$ [C.D.L.U. 2013]

6. Show that $\sin x + \cos x = 1 + x - \dfrac{1}{2!}x^2 - \dfrac{1}{3!}x^3 + \dfrac{1}{4!}x^4(\sin\theta x + \cos\theta x)$.

   [M.D.U. 2013, 12]

Working rule for finding Maxima-Minima of a function $f(x, y)$

1. Find f_x and f_y.
2. Solve the equation $f_x = 0$ and $f_y = 0$ simultaneously for x and y. Let the pair of points be $(x_1, y_1), (x_2, y_2), \ldots$
3. Let $A = f_{xx}$, $B = f_{xy}$, $C = f_{yy}$. Find the values of A, B and C for each pair of points, obtained in step 2.
4. (i) If for a pair of value (say (x_1, y_1)), $AC - B^2 > 0$ and $A < 0$, then $f(x, y)$ has a **maximum** for this pair and $f(x_1, y_1)$ is the maximum value of $f(x, y)$.

   (ii) If $AC - B^2 > 0$ and $A > 0$ at (x_1, y_1), then $f(x, y)$ has a **minimum** for this pair and $f(x_1, y_1)$ is the minimum value of $f(x, y)$.

   (iii) If $AC - B^2 < 0$ at (x_1, y_1), then there is neither maximum nor minimum at (x_1, y_1) and $f(x, y)$ is said to have a **saddle point** at (x_1, y_1).

   (iv) If $AC - B^2 = 0$ at (x_1, y_1), the test is inconclusive and needs further investigation.

SOLVED EXAMPLES

Example 1. *Examine for extreme value $f(x, y) = 3x^2 - y^2 + x^3$.*

[K.U. 2015, 14; M.D.U. 2014, 05]

Solution. We have $f(x, y) = 3x^2 - y^2 + x^3$

$$\therefore \quad \frac{\partial f}{\partial x} = 6x + 3x^2 = 3x(2 + x)$$

$$\frac{\partial^2 f}{\partial x^2} = 6 + 6x$$

$$\frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial^2 f}{\partial y^2} = -2$$

and
$$\frac{\partial^2 f}{\partial x \partial y} = 0.$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

i.e.,
$$3x(2 + x) = 0 \quad \Rightarrow \quad x = 0, -2 \quad \text{and} \quad -2y = 0 \quad \Rightarrow \quad y = 0$$

Thus the extreme points are (0, 0) and (– 2, 0).

At (0, 0) : $A = \frac{\partial^2 f}{\partial x^2} = 6$, $B = \frac{\partial^2 f}{\partial x \partial y} = 0$ and $C = \frac{\partial^2 f}{\partial y^2} = -2$

$$\therefore \quad AC - B^2 = -12 < 0$$

$\therefore$ (0, 0) is a saddle point.

At (– 2, 0) : $A = -6,\ B = 0,\ C = -2$

$$\therefore \quad AC - B^2 = 12 > 0 \quad \text{and} \quad A = -6 < 0$$

$\therefore$ (– 2, 0) is a point of maximum and maximum value $= f(-2, 0) = 12 - 0 + (-8) = 4$.

Example 2. *Examine the function $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$ for maxima and minima.* [M.D.U. 2012, 07]

Solution. We have $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 - 63 + 12y = 3(x^2 + 4y - 21)$$

and
$$\frac{\partial f}{\partial y} = 3y^2 - 63 + 12x = 3(y^2 + 4x - 21)$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\therefore \quad x^2 + 4y - 21 = 0 \qquad ...(1)$$

and
$$y^2 + 4x - 21 = 0 \qquad ...(2)$$

Subtracting (2) from (1), we have

$$(x^2 - y^2) + 4(y - x) = 0$$

or
$$(x - y)(x + y - 4) = 0$$

i.e.,
$$x - y = 0 \quad \text{or} \quad x + y - 4 = 0$$

$$\Rightarrow \quad x + y = 4 \quad \text{and} \quad x - y = 0$$

When $x + y = 4$ *i.e.,* $y = 4 - x$, then from (1),

$$x^2 + 16 - 4x - 21 = 0$$

or
$$x^2 - 4x - 5 = 0$$

i.e.,
$$(x - 5)(x + 1) = 0 \quad \Rightarrow \quad x = 5, -1$$

Now when $x = 5, y = -1$ and when $x = -1, y = 5$

Thus the points are (5, – 1) and (– 1, 5)

When $x - y = 0$ *i.e.,* $x = y$, then from (2),

$$y^2 + 4y - 21 = 0$$

i.e., $(y + 7)(y - 3) = 0 \Rightarrow y = -7, 3$

Now when $y = -7$, $x = -7$ and when $y = 3$, $x = 3$

Thus the points are $(-7, -7)$ and $(3, 3)$

Hence the extreme points are $(5, -1), (-1, 5), (-7, -7)$ and $(3, 3)$.

Also, $A = \dfrac{\partial^2 f}{\partial x^2} = 6x$, $B = \dfrac{\partial^2 f}{\partial x \partial y} = 12$ and $C = \dfrac{\partial^2 f}{\partial y^2} = 6y$

At $(5, -1)$: $A = 30$, $B = 12$, $C = -6$

$\therefore$ $AC - B^2 = -180 - 144 < 0$

Thus f has neither maximum nor minimum at $(5, -1)$ and so $(5, -1)$ is a saddle point.

At $(-1, 5)$: $A = -6$, $B = 12$, $C = 30$

$\therefore$ $AC - B^2 = -180 - 144 = -324 < 0$

Thus f has neither maximum nor minimum at $(-1, 5)$ and so $(1, -5)$ is a saddle point.

At $(-7, -7)$: $A = -42$, $B = 12$, $C = -42$

$\therefore$ $AC - B^2 = 1620 > 0$ and $A < 0$

Thus f has a maximum at $(-7, -7)$ and max. value $= f(-7, -7) = 784$.

At $(3, 3)$: $A = 18$, $B = 12$, $C = 18$

$\therefore$ $AC - B^2 = 324 - 144 = 180 > 0$ and $A = 18 > 0$

Thus f has a minimum at $(3, 3)$ and min. value $= f(3, 3) = -216$.

Example 3. ***Examine for maximum and minimum values of the function***

$$f(x, y) = xy(a - x - y).$$

[K.U. 2008]

Solution. We have : $f(x, y) = xy(a - x - y)$

$$= axy - x^2y - xy^2$$

$\therefore$
$$\frac{\partial f}{\partial x} = ay - 2xy - y^2$$

and
$$\frac{\partial f}{\partial y} = ax - x^2 - 2xy$$

For extreme values, $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$

$\therefore$
$$ay - 2xy - y^2 = 0 \quad \text{...(1)}$$

and
$$ax - x^2 - 2xy = 0 \quad \text{...(2)}$$

Subtracting (2) from (1), we get

$$a(y - x) - (y^2 - x^2) = 0$$

or $$(y-x)(a-y-x) = 0$$

i.e., $$y - x = 0 \quad \text{or} \quad a - y - x = 0$$

$$\Rightarrow \qquad x = y \quad \text{and} \quad x + y = a$$

Putting $x = y$ in (1), we get

$$ay - 2y^2 - y^2 = 0$$

or $$ay - 3y^2 = 0$$

i.e., $$y(a-3y) = 0 \quad \Rightarrow \quad y = 0, \frac{a}{3}$$

Now, when $y = 0, x = 0$ and when $y = \frac{a}{3}, x = \frac{a}{3}$

Thus the points are (0, 0) and $\left(\frac{a}{3}, \frac{a}{3}\right)$.

Putting $x + y = a$ i.e., $y = a - x$ in (2), we get

$$ax - x^2 - 2x(a-x) = 0 \quad \Rightarrow \quad -ax + x^2$$

i.e., $$x(x-a) = 0 \quad \Rightarrow \quad x = 0, a$$

Now, when $x = 0, y = a$ and when $x = a, y = 0$

Thus the points are $(0, a)$ and $(a, 0)$.

Hence the extreme points are (0, 0), $(a, 0)$, $(0, a)$, $\left(\frac{a}{3}, \frac{a}{3}\right)$

Also, $$A = \frac{\partial^2 f}{\partial x^2} = -2y, \quad B = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y, \quad C = \frac{\partial^2 f}{\partial y^2} = -2x$$

At (0, 0) : $$A = 0, \quad B = a \quad \text{and} \quad C = 0$$

$\therefore$ $$AC - B^2 = -a^2 < 0$$

Thus f has neither maximum nor minimum at (0, 0) and so (0, 0) is a saddle point.

At $(a, 0)$: $$A = 0, \quad B = -a, \quad C = -2a$$

$\therefore$ $$AC - B^2 = -(-a)^2 = -a^2 < 0$$

Thus f has neither maximum nor minimum at $(a, 0)$ and so $(a, 0)$ is a saddle point.

At $(0, a)$: $$A = -2a, B = -a, C = 0$$

$\therefore$ $$AC - B^2 = -(-a)^2 = -a^2 < 0$$

Thus f has neither maximum nor minimum at $(0, a)$ and so $(0, a)$ is a saddle point.

At $\left(\frac{a}{3}, \frac{a}{3}\right)$: $$A = -\frac{2}{3}a, \quad B = -\frac{a}{3} \quad \text{and} \quad C = -\frac{2}{3}a$$

$$AC - B^2 = \left(-\frac{2}{3}a\right)\left(-\frac{2}{3}a\right) - \left(-\frac{a}{3}\right)^2$$

$\therefore$

$$= \frac{4a^2}{9} - \frac{a^2}{9} = \frac{a^2}{3} > 0 \text{ and } A < 0$$

Thus f has a maximum at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$$\text{Max. value} = f\left(\frac{a}{3}, \frac{a}{3}\right) = \left(\frac{a}{3}\right)\left(\frac{a}{3}\right)\left[a - \frac{a}{3} - \frac{a}{3}\right] = \frac{a^2}{9}\left[\frac{a}{3}\right] = \frac{a^3}{27}$$

Example 4. ***Examine for extreme values*** $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

[M.D.U. 2018, 06; K.U. 2014]

Solution. We have $\quad f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

$\therefore$

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$$

and

$$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\therefore$

$$x^3 - x + y = 0 \qquad \ldots(1)$$

and

$$y^3 + x - y = 0 \qquad \ldots(2)$$

Adding (1) and (2), we have

$$x^3 + y^3 = 0$$

or

$$(x + y)(x^2 - xy + y^2) = 0$$

$\therefore$ For real x, $x + y = 0$ is the only possibility.

Putting $y = -x$ in (1), we get

$$x^3 - x - x = 0$$

$$x^3 - 2x = 0$$

$$x(x^2 - 2) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

Hence the extreme points are $(0, 0)$; $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

Now, $\quad A = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$

$$B = \frac{\partial^2 f}{\partial y \partial x} = 4$$

and $$C = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

At (0, 0) : $A = -4, \; B = 4, \; C = -4$

$\therefore$ $$AC - B^2 = 16 - 16 = 0$$

$\therefore$ At (0, 0), further investigation is required.

For small h, k and $h \neq k$, we have

$$f(h, k) - f(0, 0) = h^4 + k^4 - 2h^2 + 4hk - 2k^2$$

$$= -2(h - k)^2 < 0 \quad [\textit{Neglecting } h^4, k^4 \textit{ as } h, k \textit{ are small}]$$

For $h = k$, we have

$$f(h, k) - f(0, 0) = h^4 + h^4 - 2h^2 + 4h^2 - 2h^2$$

$$= 2h^4 > 0$$

As $f(h, k) - f(0, 0)$ does not keep the same sign for all small values of h and k, so the point (0, 0) is a saddle point.

At $(\sqrt{2}, -\sqrt{2})$: $A = 20, \; B = 4, \; C = 20$

$\therefore$ $$AC - B^2 > 0 \text{ and } A > 0$$

$\Rightarrow$ f has a minimum at $(\sqrt{2}, -\sqrt{2})$.

Minimum value $= f(\sqrt{2}, -\sqrt{2})$

$$= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2$$

$$= 4 + 4 - 4 - 8 - 4 = -8.$$

At $(-\sqrt{2}, \sqrt{2})$: $A = 20, \; B = 4, C = 20$

$\therefore$ $$AC - B^2 = 400 - 16 = 384 > 0 \text{ and } A = 20 > 0$$

$\therefore$ $f(x, y)$ has a minimum at $(-\sqrt{2}, \sqrt{2})$

Minimum value $= f(-\sqrt{2}, \sqrt{2}) = -8.$

Example 5. ***Examine for maximum and minimum values the function***

$\sin x + \sin y + \sin (x + y)$. ***[K.U. 2016; M.D.U. 2015, 0...***

Solution. Let $f(x, y) = \sin x + \sin y + \sin (x + y)$

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$\cos x + \cos(x+y) = 0 \quad \text{...(1)}$$

$$\cos y + \cos(x+y) = 0 \quad \text{...(2)}$$

Subtracting (2) from (1), we have

$$\cos x - \cos y = 0 \Rightarrow \cos x = \cos y \Rightarrow x = y$$

From (1),

$$\cos x + \cos 2x = 0$$

$$\cos x = -\cos 2x = \cos(\pi - 2x)$$

$\Rightarrow$
$$x = \pi - 2x \Rightarrow 3x = \pi$$

$\Rightarrow$
$$x = \frac{\pi}{3} \text{ and hence } y = \frac{\pi}{3}$$

$\therefore$ $P\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ is a stationary point.

Now,
$$A = \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y) = -[\sin x + \sin(x+y)]$$

$$B = \frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y)$$

$$C = \frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y) = -[\sin y + \sin(x+y)]$$

$\therefore$ At point $P\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$: $A = -\left[\sin\frac{\pi}{3} + \sin\frac{2\pi}{3}\right]$

$$= -\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = -\sqrt{3}$$

$$B = -\sin\frac{2\pi}{3} = -\sin\left(\pi - \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$C = -\left[\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right] = -\sqrt{3}$$

$$\therefore \qquad AC - B^2 = 3 - \frac{3}{4} = \frac{9}{4} > 0 \text{ and } A < 0.$$

Thus $f(x, y)$ has a maximum at P

and $$\text{maximum value} = \sin\frac{\pi}{3} + \sin\frac{\pi}{3} + \sin\frac{2\pi}{3} = \frac{3\sqrt{3}}{2}.$$

Example 6. *Show that the rectangular solid of maximum volume that can be inscribed in a given sphere is a cube.* [K.U. 2012; M.D.U. 2012, 10]

Solution. Let x, y, z be the dimensions of the rectangular solid inscribed in a sphere of diameter d.

$\therefore$ By symmetry, the diagonal of the rectangular solid is equal to the diameter of the sphere.

$$\therefore \qquad d^2 = x^2 + y^2 + z^2$$

$$\text{or} \qquad z^2 = d^2 - x^2 - y^2 \qquad ...(1)$$

Also, volume V of the rectangular solid is given by

$$V = xyz$$

$$\therefore \qquad V^2 = x^2y^2z^2 = x^2y^2(d^2 - x^2 - y^2) \qquad \text{[Using (1)]}$$

$$\text{Let} \qquad f(x, y) = d^2x^2y^2 - x^4y^2 - x^2y^4$$

$$\therefore \qquad \frac{\partial f}{\partial x} = d^2y^2 2x - 4x^3y^2 - 2xy^4$$

$$\frac{\partial f}{\partial y} = 2d^2x^2y - 2x^4y - 4x^2y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 2d^2y^2 - 12x^2y^2 - 2y^4$$

$$\frac{\partial^2 f}{\partial y \partial x} = 4xyd^2 - 8x^3y - 8xy^3$$

$$\frac{\partial^2 f}{\partial y^2} = 2d^2x^2 - 2x^4 - 12x^2y^2$$

For stationary points, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$2xy^2(d^2 - 2x^2 - y^2) = 0$$

$$2x^2y(d^2 - x^2 - 2y^2) = 0$$

$$d^2 - 2x^2 - y^2 = 0 \qquad ...(2)$$

$$d^2 - x^2 - 2y^2 = 0 \qquad ...(3)$$

$\therefore$

$$2x^2 + y^2 = d^2 \qquad ...(4)$$

$$x^2 + 2y^2 = d^2 \qquad ...(5)$$

Subtracting (5) from (4), we get

$$x^2 - y^2 = 0 \quad \Rightarrow \quad x = y$$

$\therefore$ From (4), we get

$$2x^2 + x^2 = d^2 \quad \Rightarrow \quad 3x^2 = d^2$$

$\Rightarrow$

$$x^2 = \frac{d^2}{3} \quad \Rightarrow \quad x = \frac{d}{\sqrt{3}} = y$$

From (1),

$$z^2 = d^2 - \frac{d^2}{3} - \frac{d^2}{3} = \frac{d^2}{3}$$

$\therefore$

$$z = \frac{d}{\sqrt{3}}.$$

At $x = y = \dfrac{d}{\sqrt{3}}$,

$$A = \frac{\partial^2 f}{\partial x^2} = 2d^2y^2 - 12x^2y^2 - 2y^4$$

$$= 2d^2 . \frac{d^2}{3} - 12 . \frac{d^2}{3} . \frac{d^2}{3} - 2\frac{d^4}{9}$$

$$= \frac{2d^4}{3} - \frac{12d^4}{9} - \frac{2d^4}{9} = -\frac{8d^4}{9}$$

$$B = \frac{\partial^2 y}{\partial x \partial y} = 4xyd^2 - 8x^3y - 8xy^3$$

$$= \frac{4d^4}{3} - \frac{8d^4}{9} - \frac{8d^4}{9} = -\frac{4d^4}{9}.$$

$$C = \frac{\partial^2 f}{\partial y^2} = 2d^2x^2 - 2x^4 - 12x^2y^2$$

$$= \frac{2d^4}{3} - \frac{2d^4}{9} - \frac{12d^4}{9} = -\frac{8d^4}{9}$$

Here $$AC - B^2 = \frac{64d^8}{81} - \frac{16d^8}{81} = \frac{48d^8}{81} > 0$$

and $$A = -\frac{8d^4}{9} < 0$$

$\therefore$ Volume is maximum when $x = y = z = \dfrac{d}{\sqrt{3}}$.

Hence, the solid is a cube.

Example 7. ***A rectangular box without top is to have volume 32 cubic feet. Find the dimensions of the box requiring least material for its construction.***

[K.U. 2017, 16; M.D.U. 2014, 11]

Solution. Let length of the box $= x$ feet

Breadth of the box $= y$ feet

Height of the box $= z$ feet

It is given that volume of box $= 32$ cubic feet ...(1)

$$\therefore \quad xyz = 32$$

$$\Rightarrow \quad z = \frac{32}{xy}$$

Let S be the surface area of the box (without top)

$$\therefore \quad S = xy + 2yz + 2zx$$

Putting $z = \dfrac{32}{xy}$, we have

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x$$

$$\therefore \quad S = xy + \frac{64}{x} + \frac{64}{y} \quad ...(2)$$

For extreme values, $\dfrac{\partial S}{\partial x} = 0$ and $\dfrac{\partial S}{\partial y} = 0$

Here $$\frac{\partial S}{\partial x} = y - \frac{64}{x^2}$$

$$\frac{\partial S}{\partial y} = x - \frac{64}{y^2}$$

$$A = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}$$

$$B = \frac{\partial^2 S}{\partial x \, \partial y} = 1$$

$$C = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

Putting $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$, we get

$$y - \frac{64}{x^2} = 0 \quad \Rightarrow \quad y = \frac{64}{x^2} \qquad ...(1)$$

$$x - \frac{64}{y^2} = 0 \quad \Rightarrow \quad x = \frac{64}{y^2} \qquad ...(2)$$

Putting the value of y from (1) in (2), we have

$$x = \frac{64}{(64)^2}(x^4) = \frac{x^4}{64}$$

$$\Rightarrow \qquad x\left(\frac{x^3}{64} - 1\right) = 0$$

$\Rightarrow$ either $x = 0$ or $x^3 = 64$

But $x = 0$ is not possible

$$\therefore \qquad x^3 = 64 \quad \Rightarrow \quad x = 4$$

Now when $x = 4, y = 4$

$\therefore$ (4, 4) is a stationary point.

At (4, 4) : $$A = \frac{128}{64} = 2, \; B = 1, \; C = \frac{128}{64} = 2$$

$$\therefore \qquad AC - B^2 = 4 - 1 = 3 > 0 \text{ and } A > 0$$

$\therefore$ (4, 4) is a point of minima *i.e.*, S is least at (4, 4)

$\therefore$ Dimensions of box having least surface area and thus requiring least material for construction will be

Length $= x =$ **4 ft.**

Breadth $= y =$ **4 ft.**

Height $= z = \frac{32}{xy} = \frac{32}{4 \times 4} =$ **2 ft.**

EXERCISE 7.1

Examine for maximum and minimum values of the following :

1. (*i*) $x^3 - 3axy + y^3$ [K.U. 2000]
   (*ii*) $y^2 + x^2y + x^4$
   (*iii*) $x^2 + xy + y^2 + x + y$
   (*iv*) $x^3 + y^3 - 3x - 12y + 20$ [K.U. 2013; M.D.U. 2011]
   (*v*) $x^2 + y^2 + 6x + 12$ [K.U. 2017, 15; M.D.U. 2004]
   (*vi*) $x^2y + xy^2 - 12xy$ [K.U. 2006; M.D.U. 2004]
   (*vii*) $x^3y^2(1 - x - y)$
   (*viii*) $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$
   (*ix*) $x^2y^2 - 5x^2 - 8xy - 5y^2$ [C.D.L.U. 2013]
   (*x*) $x^2 - xy + y^2 + 3x - 2y + 1$ [K.U. 2012]

2. Examine for maximum and minimum values of the functions :
   (*i*) $xy + \dfrac{a^3}{x} + \dfrac{a^3}{y}$, $a > 0$
   (*ii*) $xy + \dfrac{a^3}{x} + \dfrac{b^3}{y}$, $a > 0$, $b > 0$ [K.U. 2011]

3. (*i*) Show that $f(x, y) = 2x^4 - 3xy + y^2$ has neither a maximum nor a minimum at (0, 0).
   (*ii*) Show that the function $f(x, y) = (x - y)^2 + x^3 - y^3 + x^5$ has neither maximum nor a minimum at the origin.

4. Show that $f(x, y) = (y - x)^4 + (x - 2)^4$ has a minimum at (2, 2).

5. (*i*) Examine for maximum and minimum values of the function
   $$\cos x + \cos y + \cos(x + y).$$ [M.D.U. 2014]
   (*ii*) Examine for maximum and minimum values of the function
   $$f(x, y) = 2\sin\frac{1}{2}(x + y)\cos\frac{1}{2}(x - y) + \cos(x + y)$$ [M.D.U. 2017]

6. A rectangular box, open at the top, is to have a volume of $\dfrac{27}{2}$ cubic. ft. Find the dimensions of the box requiring least material for construction. [M.D.U. 2012, 07]

7. Show that of all triangles inscribed in a circle, the one with the maximum area is equilateral.

8. Show that of all triangles of given perimeter, the one with maximum area is equilateral. [M.D.U. 2011]

9. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ [M.D.U. 2015; K.U. 2014, 04]

10. In a plane triangle ABC, find the maximum value of cos A cos B cos C. [M.D.U. 2017]